Portfolio Choice

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Session 3

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Part 3. Mean-Variance Portfolio Theory

Preamble:

Mean/Variance Analysis Assumptions

Mean-variance is simpler

- Early researchers in finance, such as Markowitz and Sharpe, used just the mean and the variance of the return rate of an asset to describe it.
- Characterizing the prospects of a gamble with its mean and variance is often easier than using an NM utility function, so it is popular.
- But is it compatible with VNM theory?
- The answer is yes … approximately … under some conditions.

Mean-variance assumptions

- **Investors maximise expected utility of end-of-period wealth**
- **Can be shown that above implies maximise a function of expected portfolio returns and portfolio variance providing**

  - Either utility is quadratic, or
  - Portfolio returns are normally distributed (and utility is concave),
  - Consider only small risks (then approximately true)

Mean-variance: quadratic utility

Suppose utility is quadratic, $U(y) = ay–by^2$.

Expected utility is then $E[U(y)] = aE(y) - bE(y^2)$

$$= aE(y) - b\left[E(y)^2 + \mathrm{Var}(y)\right].$$

Thus, expected utility is a function of:

the mean: $E(y)$,

and the variance: $\mathrm{Var}(y)$

BUT not intuitive Utility function : increasing ARA

Mean-variance: joint normals

- Suppose all lotteries in the domain have normally distributed prized. (They need not be independent of each other).
- Any combination of such lotteries will also be normally distributed.
- The normal distribution is completely described by its first two moments.
- Therefore, the distribution of any combination of lotteries is also completely described by just the mean and the variance.
- As a result, expected utility can be expressed as a function of just these two numbers as well.

Mean-variance: small risks

- The most relevant justification for mean-variance is probably the case of small risks.
- If we consider only small risks, we may use a second order Taylor approximation of the NM utility function.
- A second order Taylor approximation of a concave function is a quadratic function with a negative coefficient on the quadratic term.
- In other words, any risk-averse NM utility function can locally be approximated with a quadratic function.
- But the expectation of a quadratic utility function can be evaluated with the mean and variance. Thus, to evaluate small risks, mean and variance are enough.

Mean-variance: small risks

- Let f : R → R be a smooth function. The Taylor approximation is

$$f(x) \approx f(x_0) + f'(x_0)\frac{(x-x_0)^1}{1!} + f''(x_0)\frac{(x-x_0)^2}{2!} + f'''(x_0)\frac{(x-x_0)^3}{3!} + \cdots$$

- So $f(x)$ can approximately be evaluated by looking at the value of f at another point x_0, and making a correction involving the first n derivatives.

- We will use this idea to evaluate $\mathrm{E}[U(y)]$.

Mean-variance: small risks

- Consider first an additive risk, i.e. $y = w+x$ where x is a zero mean random variable.
- For small variance of x, E[U(y)] is close to U(w).
- Consider the second order Taylor approximation,

$$E\left[U(w+x)\right] \approx U(w) + U'(w)E(x) + U''(w)\frac{E(x^2)}{2}$$

$$= U(w) + U''(w)\frac{\mathrm{Var}(x)}{2}.$$

- Let c be the certainty equivalent, $U(c)=E[U(w+x)]$.
- For small variance of x, c is close to w, but let us look at the first order Taylor approximation.

$$U(c) \approx U(w) + U'(w)(c-w)$$

Mean-variance: small risks

- Since E[U(w+x)] = U(c), this simplifies to

$$w - c \approx A(w)\frac{Var(x)}{2}$$

- $w - c$ is the risk premium.
- The risk premium is approximately a linear function of the variance of the additive risk, with the slope of the effect equal to half the coefficient of absolute risk.

Example: Simple mean-variance utility function

- Consider the simple Mean-Variance Utility function:

$$U = E(R_P) - \frac{1}{2} A \cdot \sigma^2(R_P)$$

$$E(R_P) = xE(R_M) + (1-x)Rf$$

$$\underset{x}{Max}\, U = \underset{x}{Max}\left[E(R_P) - \frac{1}{2} A \cdot \sigma(R_P)\right] = \underset{x}{Max}\left[Rf + x\left[E(R_M) - Rf\right] - \frac{1}{2} A \cdot x^2 \sigma^2(R_M)\right]$$

$$\Rightarrow \left[E(R_M) - Rf\right] - \frac{1}{2} 2A \cdot x\sigma^2(R_M) = 0$$

$$\Rightarrow x = \frac{E(R_M) - Rf}{A\sigma^2(R_M)}$$

Optimal position in the risky asset is inversely proportional to the level of risk aversion and the level of risk and directly proportional to the risk premium

3.1 Measuring Risk and Return

Risk and return of a 2-asset portfolio

Returns of assets 1 and 2: $R_1,\ R_2$

Weight of asset1: x

Return of the portfolio: R_p

$$E(R_P) = E(xR_1 + (1-x)R_2) = xE(R_1) + (1-x)E(R_2)$$

$$V(R) = E\left[(R - E(R))^2\right] = E(R^2) - (E(R))^2$$

$$\sigma(R) = \sqrt{V(R)}$$

Risk and return of a 2-asset portfolio

$$V(R_p) = E(R_P^{\ 2}) - (E(R_P))^2 = x^2 V(R_1) + (1-x)^2 V(R_2) + 2x(1-x)Cov(R_1, R_2)$$

$$V(R_p) = x^2 V(R_1) + (1-x)^2 V(R_2) + 2x(1-x)\boldsymbol{\sigma}(R_1)\boldsymbol{\sigma}(R_2)Corr(R_1, R_2)$$

$$Cov(R_1, R_2) = E\left[(R_1 - E(R_1))(R_2 - E(R_2))\right] = E\left[R_1 \times R_2\right] - E(R_1)E(R_2)$$

$$\boldsymbol{\rho} = \frac{Cov(R_1, R_2)}{\boldsymbol{\sigma}(R_1)\boldsymbol{\sigma}(R_2)}$$

Minimum-variance portfolio

$$\frac{d\mathrm{V}(R_P)}{\mathrm{d}x} = 0$$

$$\Leftrightarrow 2xV(R_1) + 2xV(R_2) - 2V(R_2) + 2\boldsymbol{\sigma}(R_1)\boldsymbol{\sigma}(R_2)Corr(R_1, R_2) - 4x\boldsymbol{\sigma}(R_1)\boldsymbol{\sigma}(R_2)Corr(R_1, R_2) = 0$$

$$x = \frac{V(R_2) - \boldsymbol{\sigma}(R_1)\boldsymbol{\sigma}(R_2)Corr(R_1, R_2)}{V(R_1) + V(R_2) - 2\boldsymbol{\sigma}(R_1)\boldsymbol{\sigma}(R_2)Corr(R_1, R_2)}$$

Example

Consider 2 stocks, A and B:

State	Probabilities	A	B
1	¼	0%	30%
2	¼	5%	20%
3	¼	15%	0%
4	¼	20%	-10%

$$E(R_A)=\frac{1}{4}(0\%+5\%+15\%+20\%)=10\%=E(R_B)$$

$$V(R_A)=\frac{1}{4}\left[(0\%)^2+(5\%)^2+(15\%)^2+(20\%)^2\right]-0{,}1^2=0.0063$$

$$V(R_B)=\frac{1}{4}\left[(30\%)^2+(20\%)^2+(0\%)^2+(-10\%)^2\right]-0.1^2=0.025$$

Example (cnt'd)

$$\sigma(R_A)=0{,}079$$

$$\sigma(R_B)=0{,}158$$

$$COV(R_A,R_B)=\frac{1}{4}\left[(0.00)(0.3)+(0.05)(0.2)+(0.15)(0.0)+(0.2)(-0.1)\right]-0.1^2=-0.0125$$

$$Corr(R_A,R_B)=\frac{-0.0125}{0.079\text{x}0.158}=-1$$

Example (cnt'd)

Risk and Return of the equally-weighted portfolio: $x=0,5$

State	Probabilities	1/N Portfolio
1	¼	0,5 x 0% + 0,5 x 30% = 15%
2	¼	12,5%
3	¼	7,5%
4	¼	5%

$$E(R_p) = 0.1$$

$$V(R_P)=\frac{1}{4}\left[(15\%)^2+(12.5\%)^2+(7.5\%)^2+(5\%)^2\right]-0.1^2=0.0016$$

or

$$V(R_P)=(0.5^2)(0.0063)+(0.5^2)(0.025)+2\text{x}0.5\text{x}0;5\text{x}(\text{-}1)(0.079)(0.158)=0.0016$$

$\sigma(R_P)=0.067$ $<\sigma(R_B)<\sigma(R_A)$ **Diversification effect**

Example (cnt'd)

The minimum Variance portfolio:

$$x = \frac{V(R_2) - \sigma(R_1)\sigma(R_2)Corr(R_1, R_2)}{V(R_1) + V(R_2) - 2\sigma(R_1)\sigma(R_2)Corr(R_1, R_2)}$$

$$x = \frac{0.025 - 0.079\text{x}0.158\text{x}(-1)}{0.0063 + 0.025 - 2\text{x}0.079\text{x}0.158\text{x}(-1)} = \frac{2}{3}$$

$$V(R_P) = (0.67^2)(0.0063) + (0.33^2)(0.025) + 2\text{x}(0.67)\text{x}(0.33)\text{x}(-1)(0.079)(0.158) = 0$$

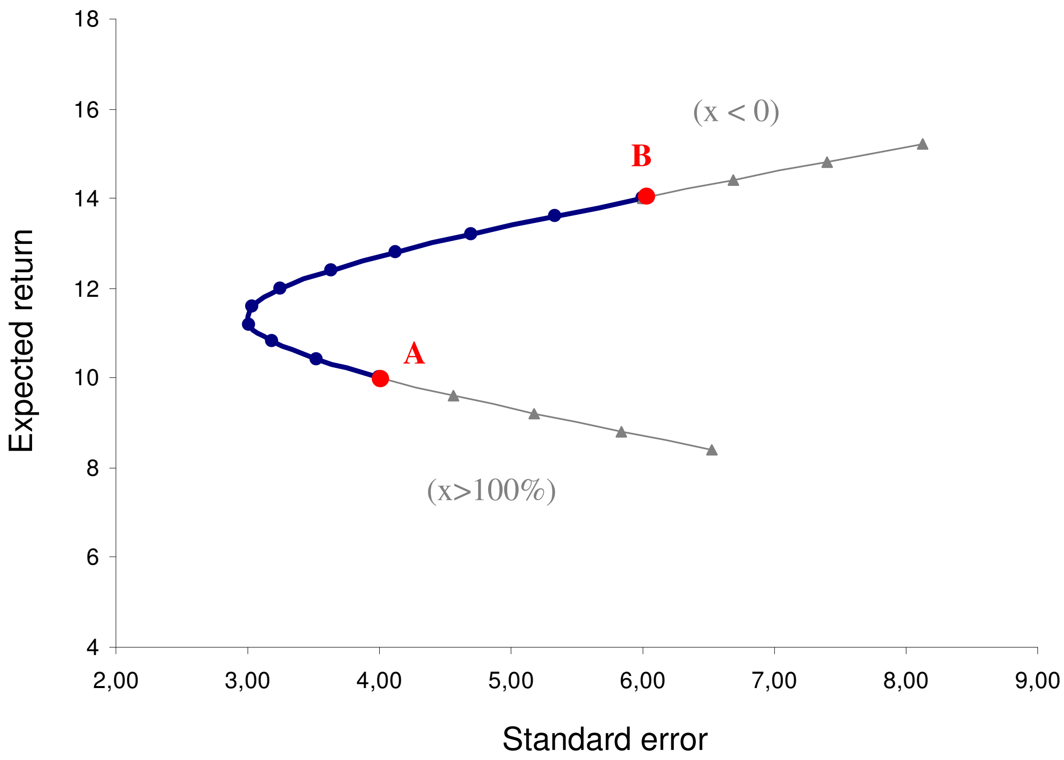
since $\rho_{R_A, R_B} = -1$

3.2 Asset Allocation with 2 Risky Assets

Portfolio characteristics and correlations

- Correlation (covariance) between 2 assets $\Rightarrow$ Diversification gain

- General case:



Portfolio characteristics and correlations (cnt'd)

- Special cases:

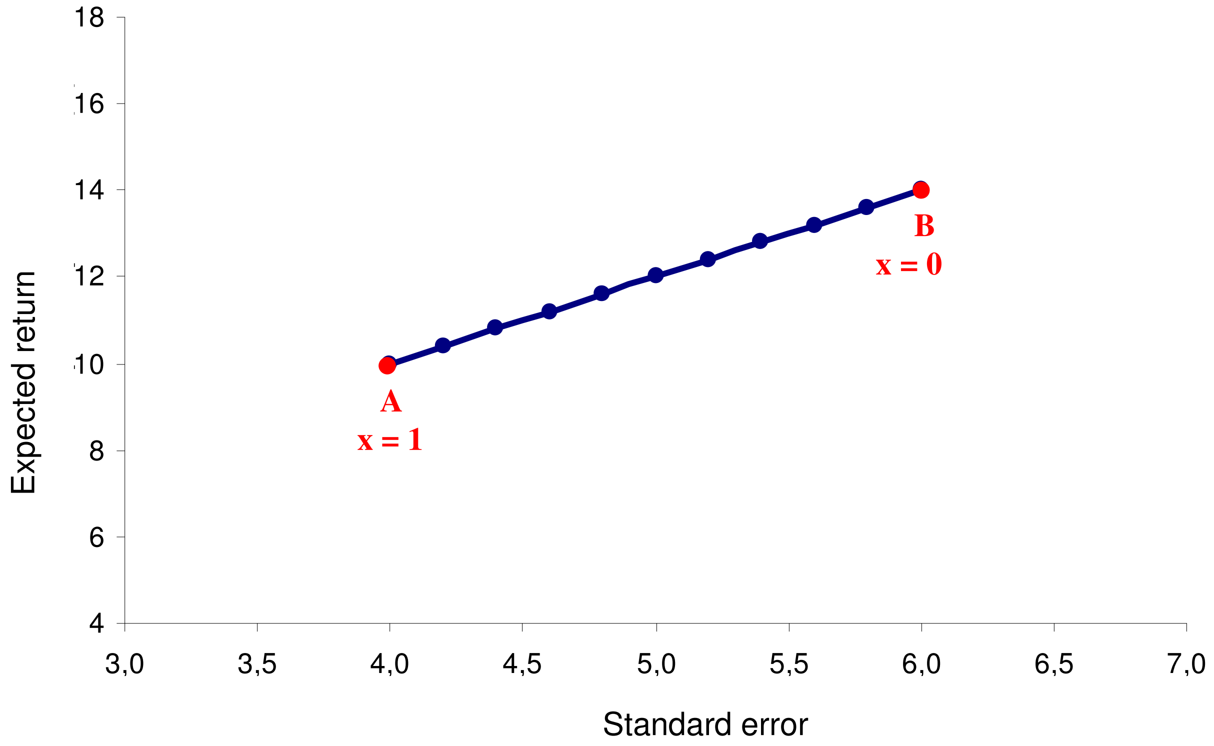
$$Corr(R_1, R_2) = 1 \quad \Rightarrow \quad V(R_p) = \left[x\sigma(R_1) + (1-x)\sigma(R_2) \right]^2$$

$$Corr(R_1, R_2) = -1 \quad \Rightarrow \quad V(R_p) = \left[x\sigma(R_1) - (1-x)\sigma(R_2) \right]^2$$

$$Corr(R_1, R_2) = 0 \quad \Rightarrow \quad V(R_p) = x^2 V(R_1) + (1-x)^2 V(R_2)$$

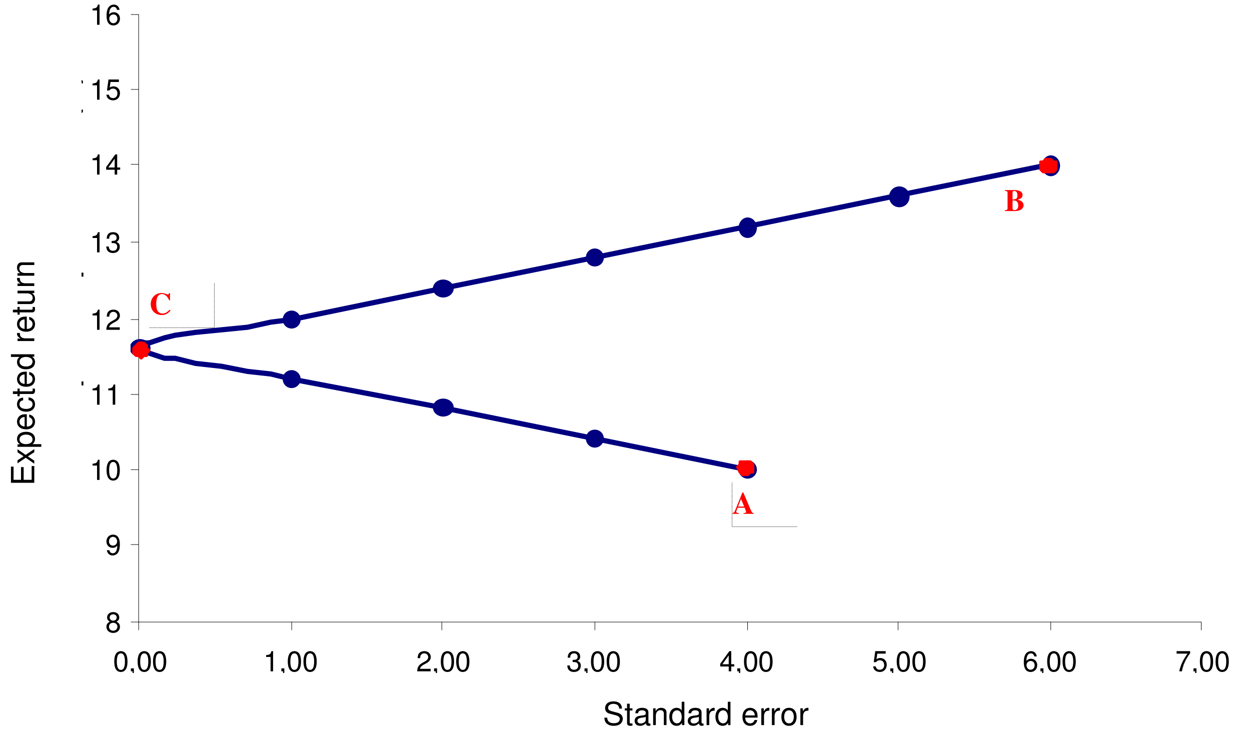
No diversification gain

$$Corr(R_1, R_2) = 1$$



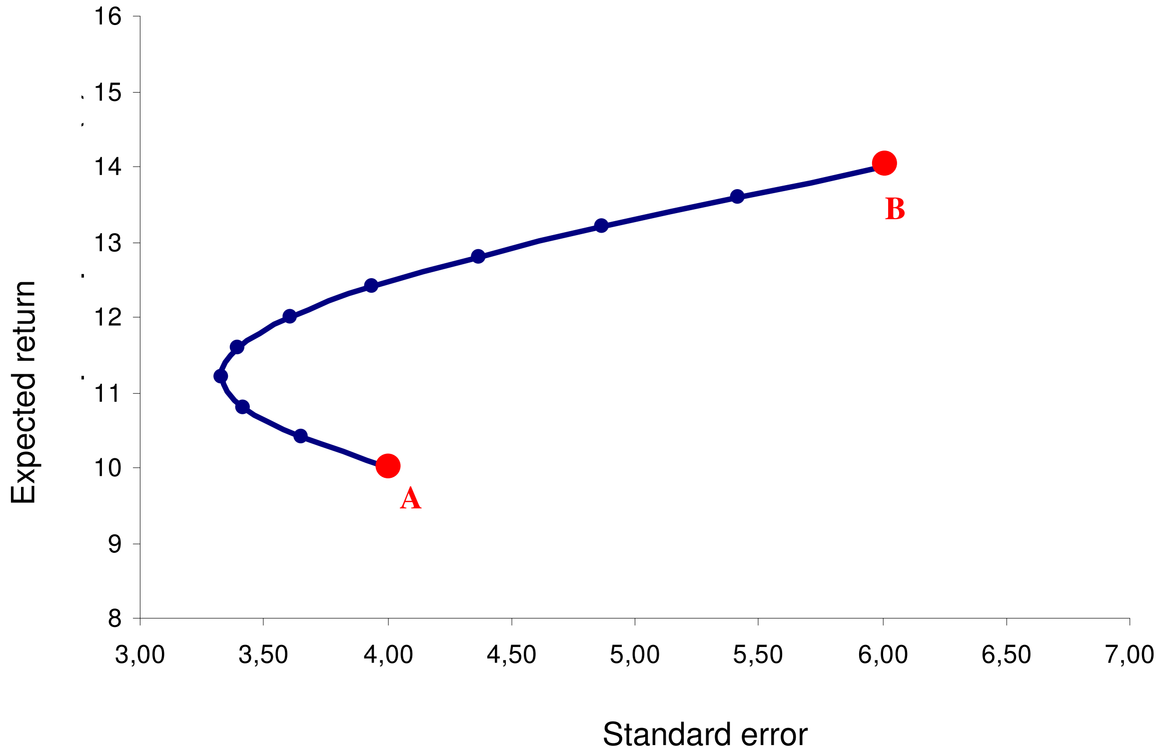
Full diversification gain

$$Corr(R_1, R_2) = -1$$

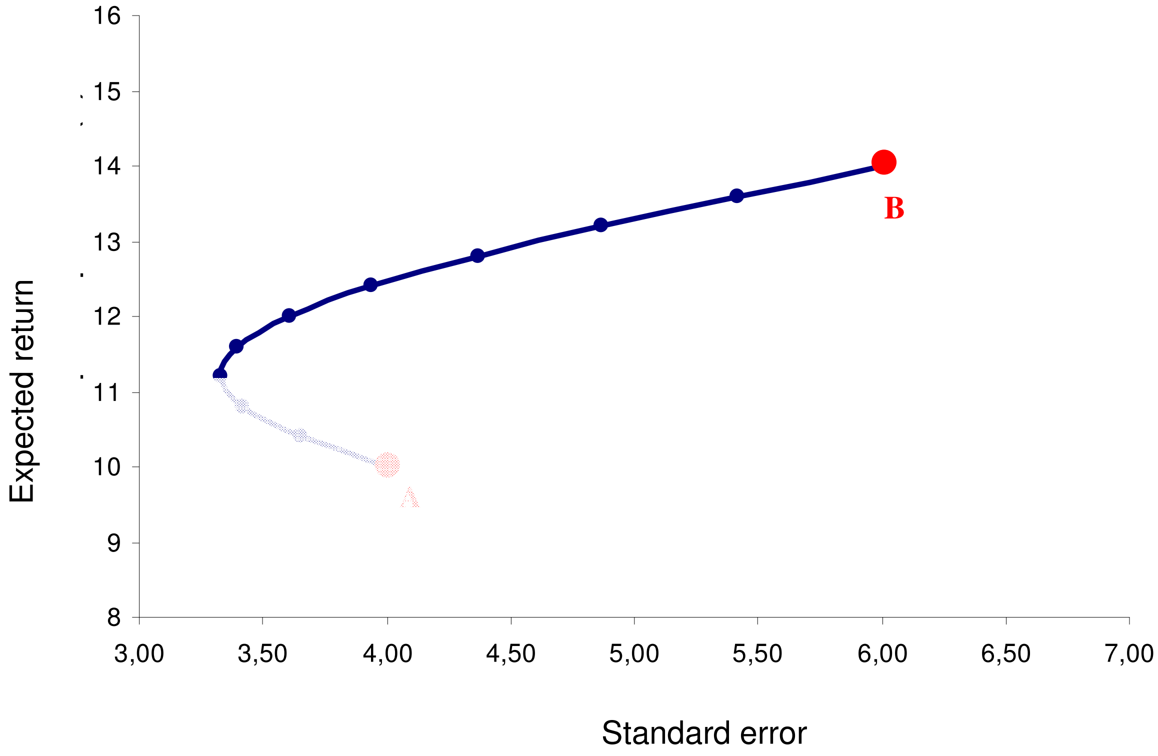


Partial diversification gain

$$Corr(R_1, R_2) = 0$$



Efficient portfolios of risky assets



3.3 Introducing a Risk Free Asset and the Tobin's Separation Theorem

Introducing borrowing and lending: Risk free asset

- You are now allowed to borrow and lend at the risk free rate Rf while still investing in any SINGLE ‘risky bundle’ on the efficient frontier.

- For each SINGLE risky bundle, this gives a new set of risk return combination known as the ‘transformation line’.

- The risk-return combination is a straight line (for each single risky bundle) - transformation line.

- You can be anywhere you like on this line.

The risk free asset

- Consider a risk-free asset (e.g. T-bill rate)

$$\begin{cases} R_f \\ \sigma_{R_f} = 0 \\ Corr(R_1, R_f) = 0 \end{cases}$$

- Consider a portfolio with a risky asset 1 and a risk-free asset:

$$E(R_P) = xE(R_1) + (1-x)R_F$$
$$V(R_P) = x^2 V(R_1)$$

The Capital Market Line

- From the precedent mean and variance expressions, we obtain:

$$E(R_P) = R_F + x\left[E(R_1) - R_F\right]$$

$$x = \frac{\sigma(R_P)}{\sigma(R_1)}$$

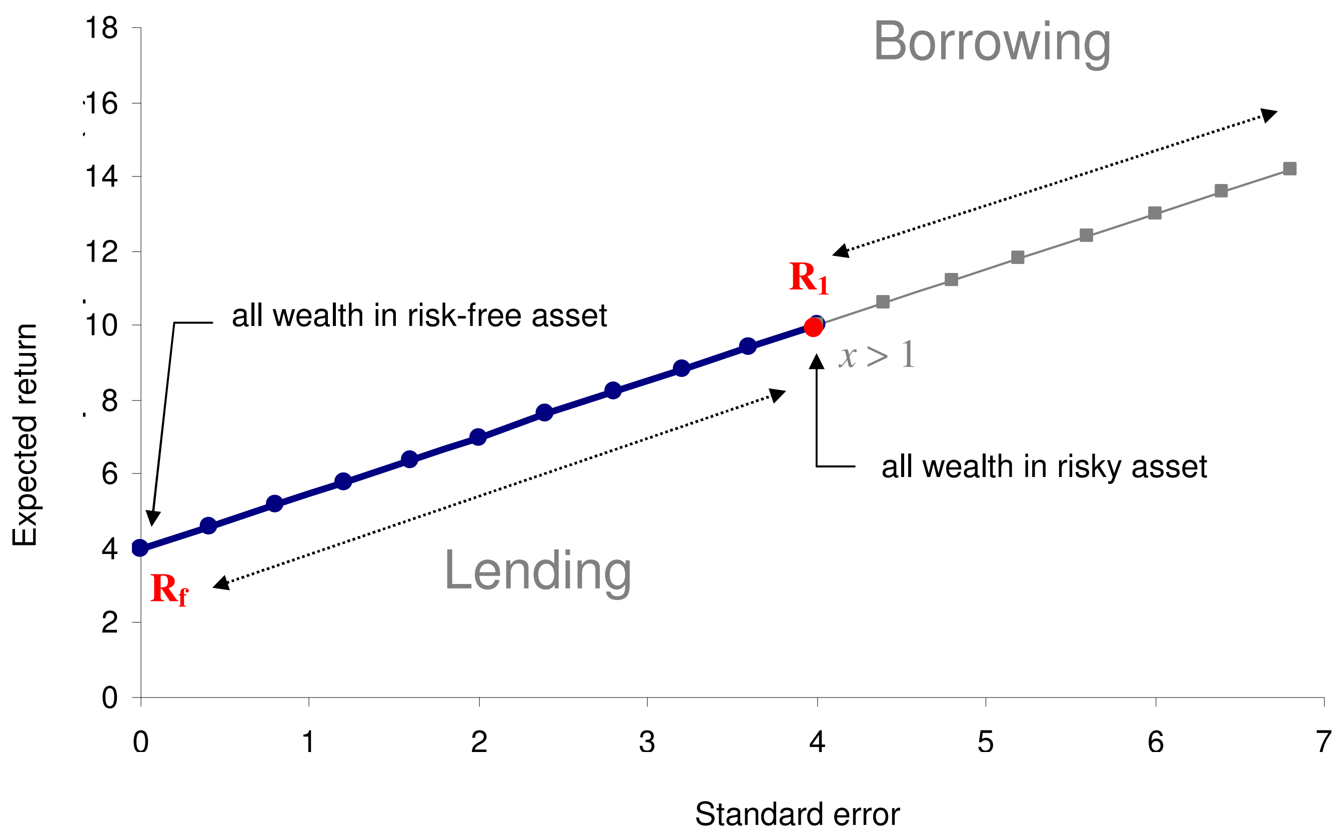
- The Capital Market Line (CML):

$$E(R_P) = E(R_f) + \frac{E(R_1) - Rf}{\sigma(R_1)}\sigma(R_P)$$

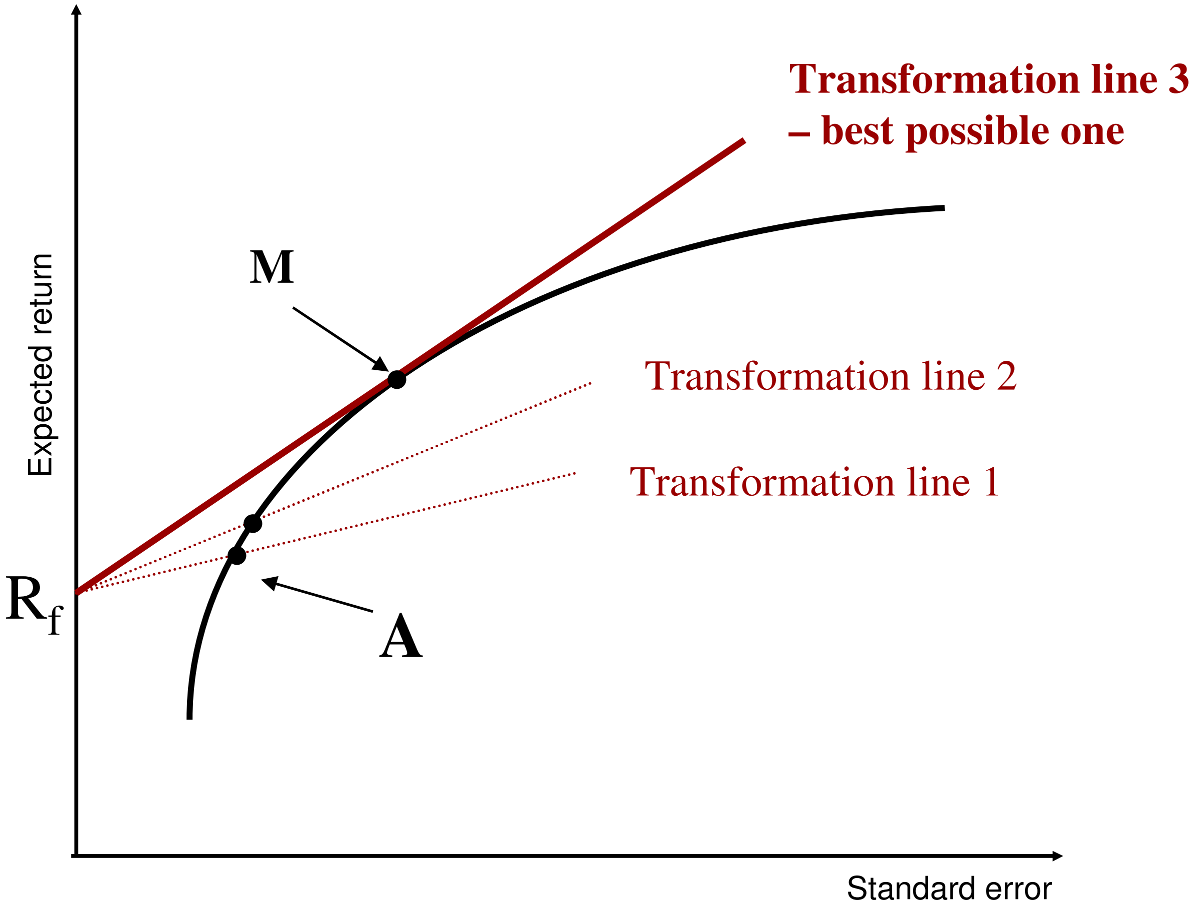
intercept — $E(R_f)$

Slope — $\frac{E(R_1) - Rf}{\sigma(R_1)}$

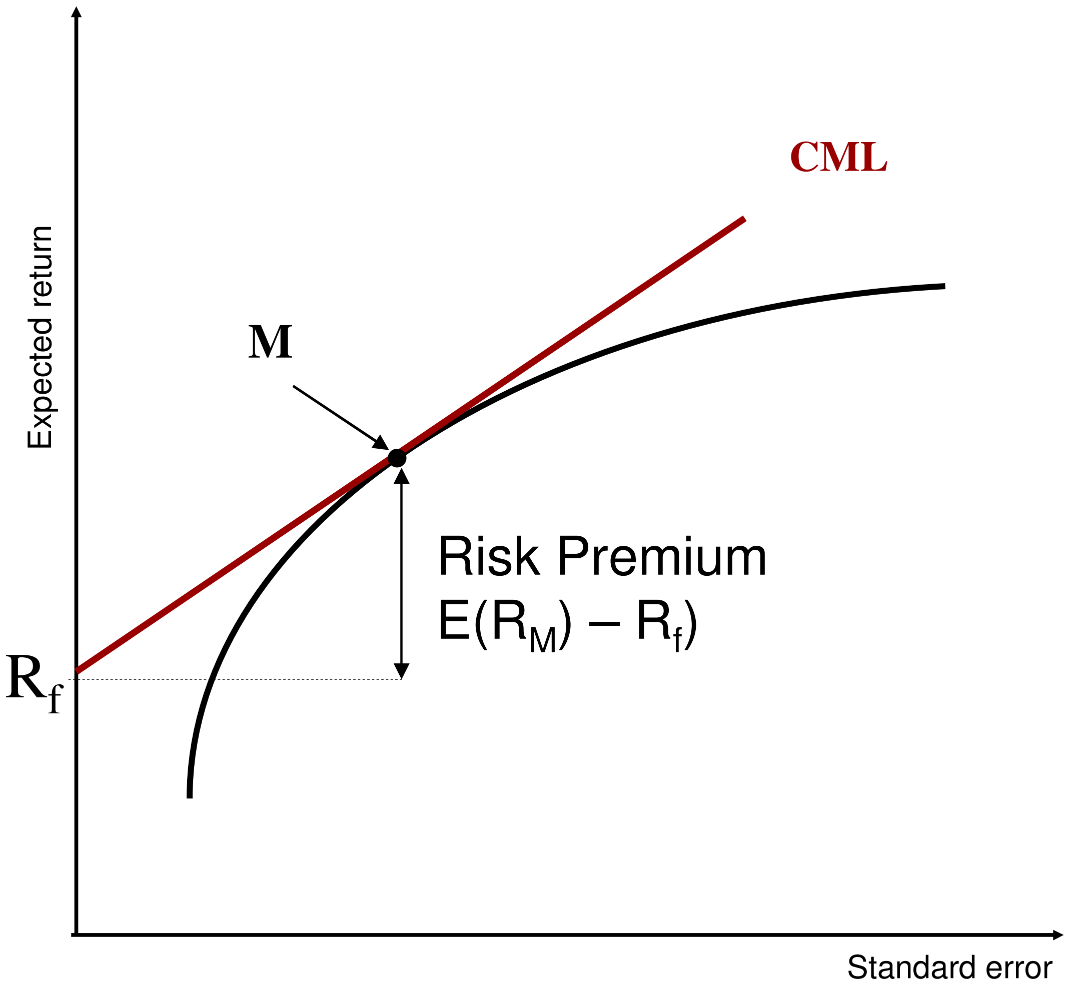
The Capital Market Line with one risky asset



The Capital Market Line with N risky assets - Best transformation line



The Capital Market Line



The CML and the separation theorem

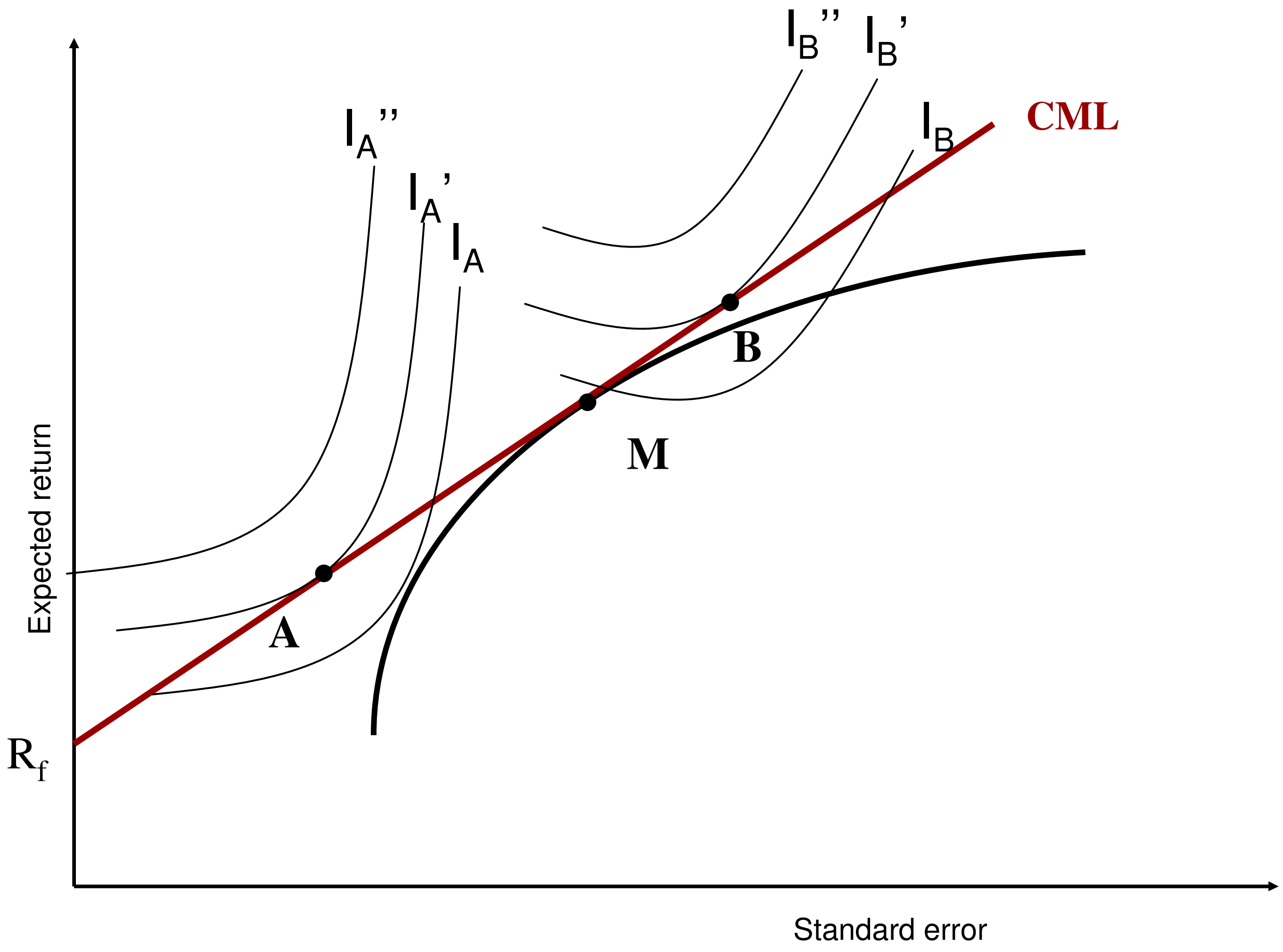
- The CML dominates all other possible portfolios
- An agent invests along the CML (where ? ⇒ risk preferences)
- James Tobin's separation theorem:

- Invest in a risky portfolio (optimal combination of risky securities)

- Borrow-lend at the risk-free rate

- Depending on your attitude toward risk: how much lend or borrow

Optimal portfolio choice



From the CML to the SML

CML: $$E(R_P) = E(R_f) + \frac{E(R_M) - Rf}{\sigma(R_M)}\sigma(R_P)$$

$$E(R_P) = E(R_f) + \frac{\sigma(R_P)}{\sigma(R_M)}\left[E(R_M) - Rf\right]$$

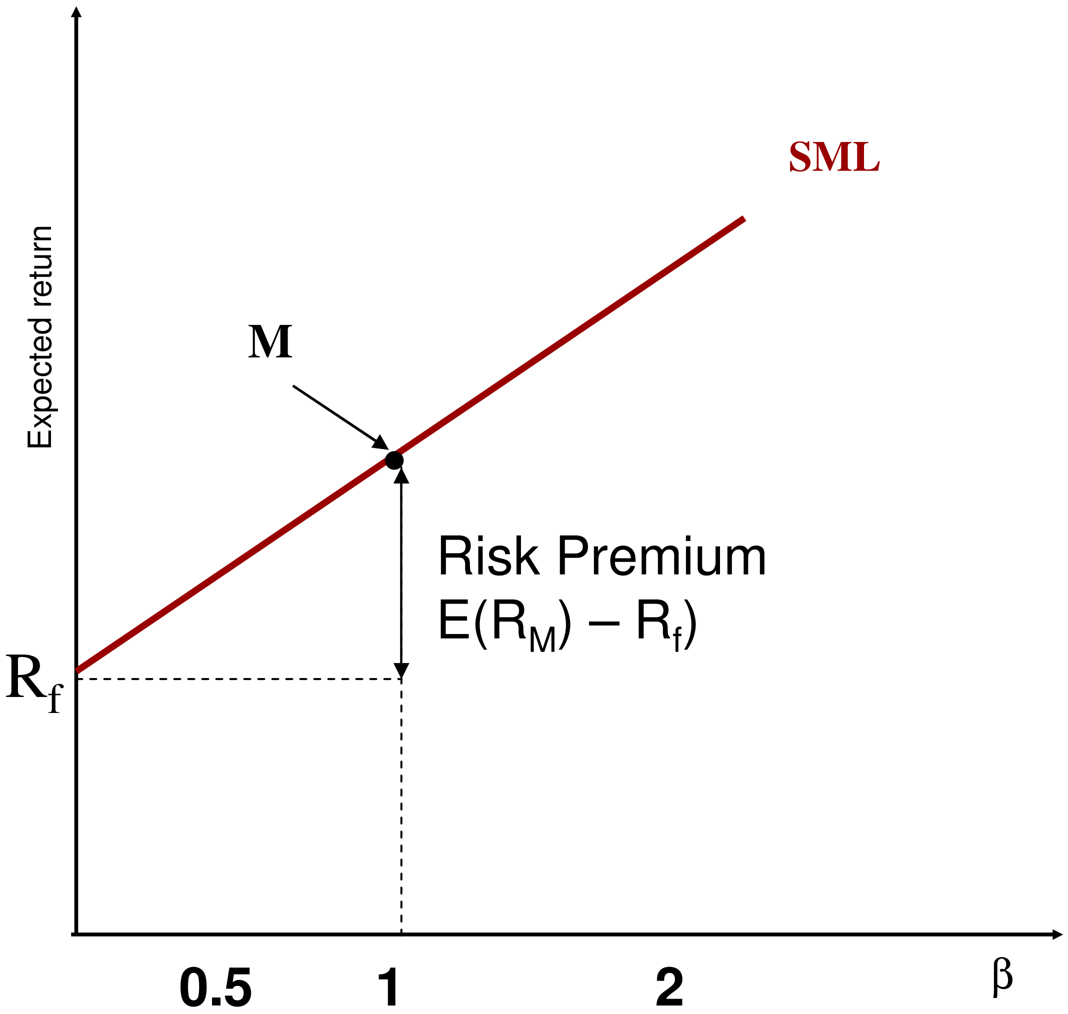
$$E(R_P) = E(R_f) + \frac{\sigma(R_P)\sigma(R_M)}{\sigma(R_M)^2}\left[E(R_M) - Rf\right]$$

Note that: $\rho_{R_M,R_p} = \frac{Cov(R_M,R_p)}{\sigma(R_M)\sigma(R_p)}$ and since: $\rho_{R_M,R_p} = 1$

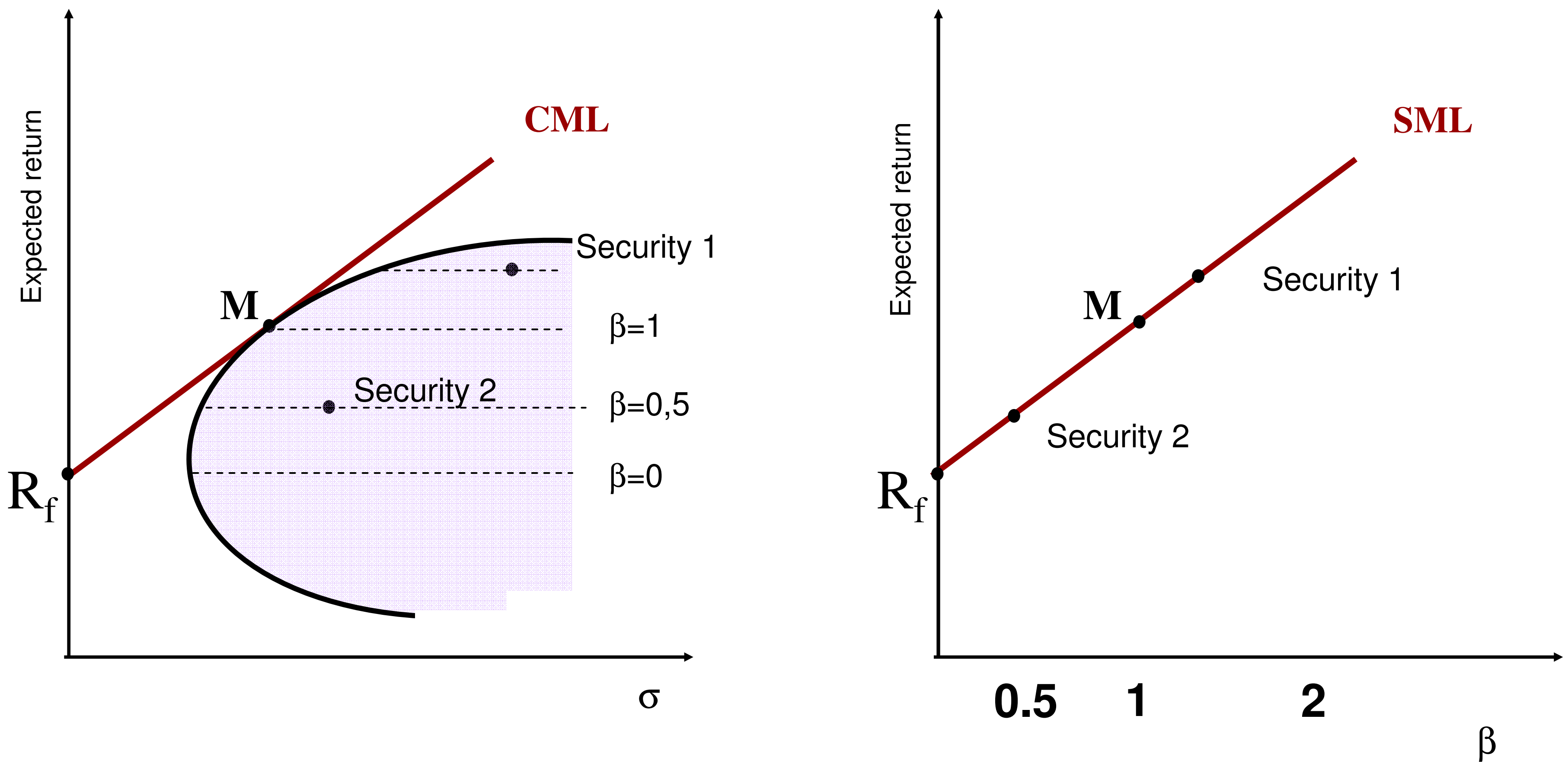
$$E(R_P) = E(R_f) + \frac{Cov(R_M,R_p)}{\sigma(R_M)^2}\left[E(R_M) - Rf\right]$$

Security Market Line: $E(R_P) = R_f + \beta\left[E(R_M) - Rf\right]$

The Security Market Line (SML)



CML vs SML


Expected return
CML
Security 1
M
β=1
Security 2
β=0,5
β=0
Rf
σ
Expected return
SML
Security 1
M
Security 2
Rf
0.5
1
2
β

CML vs SML

- The CML: combination the market portfolio and the riskless asset
- Portfolios on the CML are efficient portfolios
- Any portfolio on the CML has a correlation of 1 with the market portfolio
- CML is applicable only to an investor's efficient portfolio
- SML is applicable to any security, asset or portfolio (CAPM World)
- In the CML, risk is measured by σ
- In the SML, risk is measured by β

3.4 Asset Allocation with N risky Assets

Efficient portfolios

• **Efficient portfolios are those that maximize the expected return for a given level of expected risk**

• **or minimize the risk for a given level of expected return**

• **Two kinds of efficient portfolios:**

- only risky assets

- both risky assets and a riskless-asset (separation theorem)

• **Suppose stable expected-returns and VCV matrix**

Some notations

$$R = \begin{bmatrix} E(R_1) \\ E(R_2) \\ \vdots \\ E(R_N) \end{bmatrix} \qquad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix} \qquad \Omega = \begin{bmatrix} \boldsymbol{\sigma}_{11} \cdots \boldsymbol{\sigma}_{1N} \\ \vdots \ (\boldsymbol{\sigma}_{ij}) \ \vdots \\ \boldsymbol{\sigma}_{N1} \cdots \boldsymbol{\sigma}_{NN} \end{bmatrix}$$

$$E(R_P) = R^T X = X^T R$$

$$V(R_P) = X^T \Omega X$$

Efficient frontier with a riskless asset

- Recall the transformation lines and CML: Maximise the slope

$$\text{Max}\,\Theta = \frac{E(R_M) - Rf}{\boldsymbol{\sigma}(R_M)} \qquad \text{u.c} \qquad \sum_{i=1}^{N} x_i = 1$$

set $\quad Rf = \sum_{i=1}^{N} x_i Rf$

$$\text{Max}\,\Theta = \frac{\sum_{i=1}^{N} x_i (R_i - R_f)}{\sqrt{\sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \boldsymbol{\sigma}_{ij}}} = X^T (R - Rf)\left[X^T \Omega X\right]^{-1/2}$$

$$\frac{d\Theta}{dX} = (R - \overline{R_f})\left[X^T \Omega X\right]^{-1/2} + \left[X^T (R - \overline{R_f})\right](-\frac{1}{2}) 2\Omega X \left[X^T \Omega X\right]^{-3/2} = 0$$

Efficient frontier with a riskless asset (cnt'd)

multiplying by $\left[X^T \Omega X\right]^{1/2}$

$$(R-\overline{R_f})+\left[X^T(R-\overline{R_f})\right](-\frac{1}{2})2\Omega X\left[X^T \Omega X\right]^{-1}=0$$

Define (SR) $\lambda=\left[X^T(R-\overline{R_f})\right]\left[X^T \Omega X\right]^{-1}$

$$(R-\overline{R}_f)-\lambda\Omega X=0$$

$$\lambda X=Z$$

$$(R-\overline{R}_f)=\Omega Z$$

Efficient frontier with a riskless asset (cnt'd)

$$\begin{cases} R_1 - R_f = Z_1\sigma_{11} + Z_2\sigma_{12} + \ldots + Z_N\sigma_{1N} \\ R_2 - R_f = Z_1\sigma_{21} + Z_2\sigma_{22} + \ldots + Z_N\sigma_{2N} \\ \vdots \\ R_N - R_f = Z_1\sigma_{N1} + Z_2\sigma_{N2} + \ldots + Z_N\sigma_{NN} \end{cases}$$

$$X_i = \frac{Z_i}{\sum_{i=1}^{N} Z_i}$$

Since: $\sum_{i=1}^{N} X_i\lambda = \sum_{i=1}^{N} Z_i \Rightarrow \lambda = \sum_{i=1}^{N} Z_i$

Then we can calculate: $E(R_M)$ and $\sigma(R_M)$

and the CML $E(R_P) = R_f + \frac{E(R_1) - Rf}{\sigma(R_1)}\sigma(R_P)$

Efficient frontier with no riskless asset

$$\text{Min}\frac{1}{2}X^T\Omega X \qquad \text{u.c} \qquad \begin{cases} X^T\bar{R}=E(R_P) \\ X^T\bar{1}=1 \end{cases}$$

$$£=\frac{1}{2}X^T\Omega X+\lambda\left[E(R_P)-X^T\bar{R}\right]+\gamma\left[1-X^T\bar{1}\right]$$

$$\begin{cases} \frac{d£}{dX}=\Omega X-\lambda\bar{R}-\gamma\bar{1}=0 \\ \frac{d£}{d\lambda}=E(R_P)-X^T\bar{R}=0 \\ \frac{d£}{d\gamma}=1-X^T\bar{1}=0 \end{cases}$$

$$X=\Omega^{-1}\left(\lambda\bar{R}+\gamma\bar{1}\right)$$

Efficient frontier with no riskless asset (Cnt'd)

$$
\begin{cases}
\bar{R}^T X = \bar{R}^T \Omega^{-1}\left(\lambda \bar{R} + \gamma \bar{1}\right) = E(R_P) \\
\\
\bar{1}^T X = \bar{1}^T \Omega^{-1}\left(\lambda \bar{R} + \gamma \bar{1}\right) = 1
\end{cases}
$$

We define

$$
\begin{aligned}
A &= \bar{1}^T \Omega^{-1} \bar{R} = \bar{R}^T \Omega^{-1} \bar{1} \\
B &= \bar{R}^T \Omega^{-1} \bar{R} \\
C &= \bar{1}^T \Omega^{-1} \bar{1} \\
D &= BC - A^2
\end{aligned}
$$

$$
\lambda = \frac{CE(R_P) - A}{D}
$$

$$
\gamma = \frac{B - AE(R_P)}{D}
$$

Efficient frontier with no riskless asset (Cnt'd)

Recall: $X = \Omega^{-1}\left(\lambda\bar{R} + \gamma\bar{1}\right)$

$X = g + hE(R_P)$ with $g = \dfrac{B\Omega^{-1}\bar{1} - A\Omega^{-1}\bar{R}}{D}$

$$h = \frac{C\Omega^{-1}\bar{R} - A\Omega^{-1}\bar{1}}{D}$$

Portfolio *g* represent the optimal weights of a portfolio with $E(R_P) = 0$

$X = g + h$ represents the optimal weights of a portfolio with $E(R_P) = 1$

Characterization of frontier portfolios

- The entire set of frontier portfolios can be generated by 2 efficient portfolios, e.g. *g* and *g*+*h*
- A combination of efficient portfolios would be efficient too

⇒To find a efficient portfolio with $E(R_1)$

⇒ Build a portfolio with $\left[1-E(R_1)\right]$ of *g* and $E(R_1)$ of *g*+*h*

⇒ The structure of the portfolio is: $\left[1-E(R_1)\right]g+E(R_1)(g+h)=g+hE(R_1)$

3.5 Portfolio Diversification

Diversification and Portfolio risk

- **Market risk**

  - Systematic or Nondiversifiable (Business cycle, geopolitics, etc.)

- **Firm-specific risk**

  - Diversifiable or nonsystematic (Firm specific factors: management, sector, loss of a patent , etc.)

- **Total risk = Systematic risk + idiosyncratic risk**

Can disappear with diversification

Power of Diversification

- **As the number of assets (N) in the portfolio increases, the SD (total riskiness) falls**

- **Assumptions:**

  - All assets have the same variance : $\sigma_i^2 = \sigma^2$

  - All assets have the same covariance : $\sigma_{ij} = \rho\ \sigma^2$

  - Invest equally in each asset (i.e. 1/N)

Power of Diversification

- General formula for calculating the portfolio variance

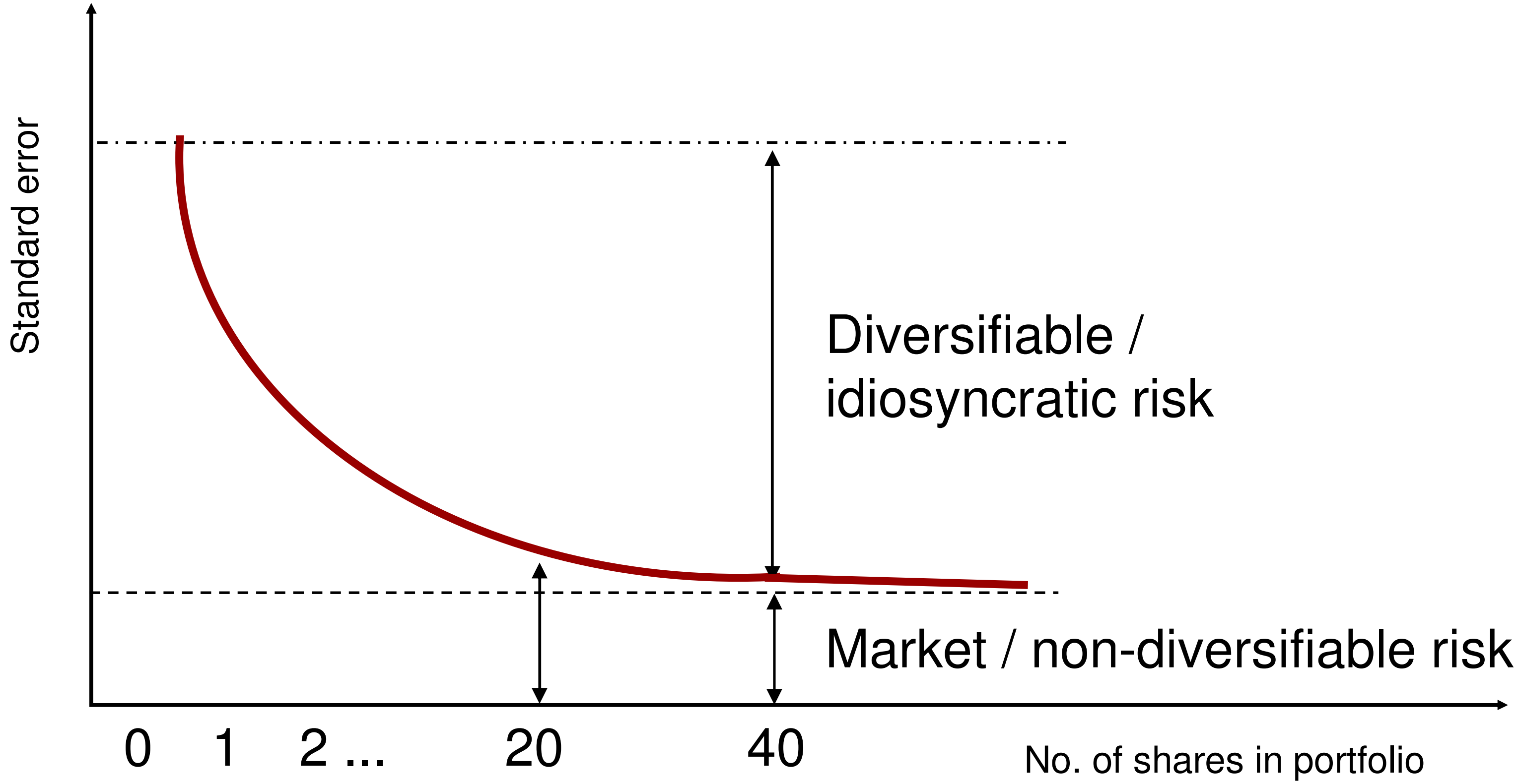
$$\sigma^2_p = \Sigma\, w_i^2\, \sigma_i^2 + \Sigma\Sigma\, w_i w_j\, \sigma_{ij}$$

- Formula with assumptions imposed

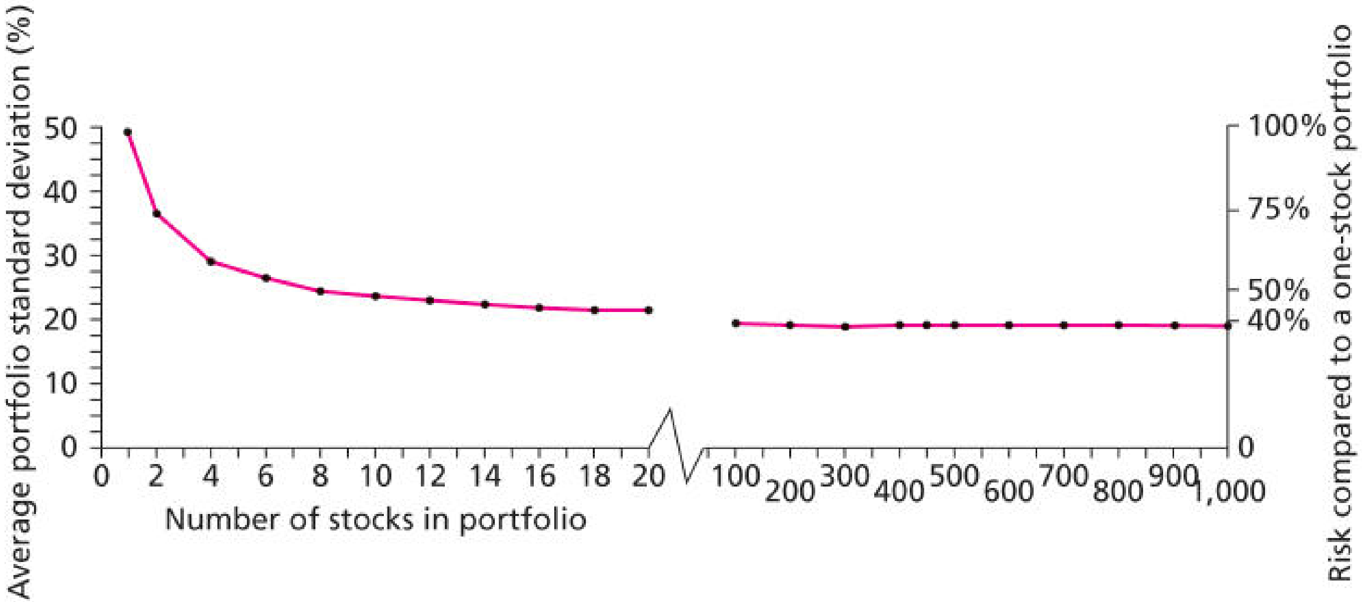
$$\sigma^2_p = (1/N)\, \sigma^2 + ((N-1)/N)\, \rho\sigma^2$$

  - If N is large, (1/N) is small and ((N-1)/N) is close to 1.
  - Hence : $\sigma^2_p \approx \rho\, \sigma^2$
  - Portfolio risk is ‘covariance risk’.

Random selection of stocks



Portfolio risk as a function of number of securities



Source: BKM (2007) from Statman (1987)

Thank you for your attention…

See you next week