Applied Econometric Time Series

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Session 3

Further development and analysis of the classical linear regression model

Generalising the Simple Model to Multiple Linear Regression

- Before, we have used the model

$$y_t = \alpha + \beta x_t + u_t \quad t = 1,2,...,T$$

- But what if our dependent (y) variable depends on more than one independent variable?

  For example the number of cars sold might plausibly depend on

  1. the price of cars
  2. the price of public transport
  3. the price of petrol
  4. the extent of the public's concern about global warming

- Similarly, stock returns might depend on several factors.
- Having just one independent variable is no good in this case - we want to have more than one x variable. It is very easy to generalise the simple model to one with k-1 regressors (independent variables).

Multiple Regression and the Constant Term

- Now we write

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \ldots + \beta_k x_{kt} + u_t \quad , t=1,2,\ldots,T$$

- Where is x_1? It is the constant term. In fact the constant term is usually represented by a column of ones of length T:

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

β_1 is the coefficient attached to the constant term (which we called α before).

Different Ways of Expressing the Multiple Linear Regression Model

- We could write out a separate equation for every value of t:

$$y_1 = \beta_1 + \beta_2 x_{21} + \beta_3 x_{31} + \ldots + \beta_k x_{k1} + u_1$$
$$y_2 = \beta_1 + \beta_2 x_{22} + \beta_3 x_{32} + \ldots + \beta_k x_{k2} + u_2$$
$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$
$$y_T = \beta_1 + \beta_2 x_{2T} + \beta_3 x_{3T} + \ldots + \beta_k x_{kT} + u_T$$

- We can write this in matrix form

$$y = X\beta + u$$

where y is $T \times 1$
X is $T \times k$
β is $k \times 1$
u is $T \times 1$

Inside the Matrices of the Multiple Linear Regression Model

- e.g. if k is 2, we have 2 regressors, one of which is a column of ones:

$$\underset{T\times 1}{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix}} = \underset{T\times 2}{\begin{bmatrix} 1 & x_{21} \\ 1 & x_{22} \\ \vdots & \vdots \\ 1 & x_{2T} \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}} + \underset{T\times 1}{\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_T \end{bmatrix}}$$

- Notice that the matrices written in this way are conformable.

How Do We Calculate the Parameters (the β) in this Generalised Case?

- Previously, we took the residual sum of squares, and minimised it w.r.t. α and β.
- In the matrix notation, we have

$$\hat{u} = \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \vdots \\ \hat{u}_T \end{bmatrix}$$

- The RSS would be given by

$$\hat{u}'\hat{u} = \begin{bmatrix} \hat{u}_1 & \hat{u}_2 & \dots & \hat{u}_T \end{bmatrix} \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \\ \\ \hat{u}_T \end{bmatrix} = \hat{u}_1^2 + \hat{u}_2^2 + \ldots + \hat{u}_T^2 = \sum \hat{u}_t^2$$

The OLS Estimator for the Multiple Regression Model

- In order to obtain the parameter estimates, β_1, β_2,..., β_k, we would minimise the RSS with respect to all the βs.

- It can be shown that

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_k \end{bmatrix} = (X'X)^{-1}X'y$$

Calculating the Standard Errors for the Multiple Regression Model

- Check the dimensions: $\hat{\beta}$ is $k \times 1$ as required.
- But how do we calculate the standard errors of the coefficient estimates?
- Previously, to estimate the variance of the errors, σ^2, we used $s^2 = \frac{\sum \hat{u}_t^2}{T-2}$.
- Now using the matrix notation, we use $s^2 = \frac{\hat{u}'\hat{u}}{T-k}$
- where k = number of regressors. It can be proved that the OLS estimator of the variance of $\hat{\beta}$ is given by the diagonal elements of $s^2(X'X)^{-1}$, so that the variance of $\hat{\beta}_1$ is the first element, the variance of $\hat{\beta}_2$ is the second element, and …, and the variance of $\hat{\beta}_k$ is the k^{th} diagonal element.

Calculating Parameter and Standard Error Estimates for Multiple Regression Models: An Example

- Example: The following model with k=3 is estimated over 15 observations:

$$y = \beta_1 + \beta_2 x_2 + \beta_3 x_3 + u$$

and the following data have been calculated from the original X's.

$$(X'X)^{-1} = \begin{bmatrix} 2.0 & 3.5 & -1.0 \\ 3.5 & 1.0 & 6.5 \\ -1.0 & 6.5 & 4.3 \end{bmatrix}, (X'y) = \begin{bmatrix} -3.0 \\ 2.2 \\ 0.6 \end{bmatrix}, \hat{u}'\hat{u} = 10.96$$

Calculate the coefficient estimates and their standard errors.

- To calculate the coefficients, just multiply the matrix by the vector to obtain $(X'X)^{-1}X'y$
- To calculate the standard errors, we need an estimate of σ^2.

$$s^2 = \frac{RSS}{T-k} = \frac{10.96}{15-3} = 0.91$$

Calculating Parameter and Standard Error Estimates for Multiple Regression Models: An Example (cont'd)

- The variance-covariance matrix of $\hat{\beta}$ is given by

$$s^2(X'X)^{-1} = 0.91(X'X)^{-1} = \begin{bmatrix} 1.83 & 3.20 & -0.91 \\ 3.20 & 0.91 & 5.94 \\ -0.91 & 5.94 & 3.93 \end{bmatrix}$$

- The variances are on the leading diagonal:

$$Var(\hat{\beta}_1) = 1.83 \qquad SE(\hat{\beta}_1) = 1.35$$
$$Var(\hat{\beta}_2) = 0.91 \Leftrightarrow SE(\hat{\beta}_2) = 0.96$$
$$Var(\hat{\beta}_3) = 3.93 \qquad SE(\hat{\beta}_3) = 1.98$$

- We write: $\hat{y} = 1.10 - 4.40x_{2t} + 19.88x_{3t}$
$$(1.35) \quad (0.96) \quad (1.98)$$

Testing Multiple Hypotheses: The *F*-test

- We used the *t*-test to test single hypotheses, i.e. hypotheses involving only one coefficient. But what if we want to test more than one coefficient simultaneously?

- We do this using the *F*-test. The *F*-test involves estimating 2 regressions.

- The unrestricted regression is the one in which the coefficients are freely determined by the data, as we have done before.

- The restricted regression is the one in which the coefficients are restricted, i.e. the restrictions are imposed on some βs.

The *F*-test: Restricted and Unrestricted Regressions

- Example

  The general regression is

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 x_{4t} + u_t \qquad (1)$$

- We want to test the restriction that $\beta_3+\beta_4 = 1$ (we have some hypothesis from theory which suggests that this would be an interesting hypothesis to study). The unrestricted regression is (1) above, but what is the restricted regression?

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 x_{4t} + u_t \quad s.t.\ \beta_3+\beta_4 = 1$$

- We substitute the restriction ($\beta_3+\beta_4 = 1$) into the regression so that it is automatically imposed on the data.

$$\beta_3+\beta_4 = 1 \Rightarrow \beta_4 = 1- \beta_3$$

The *F*-test: Forming the Restricted Regression

$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + (1-\beta_3) x_{4t} + u_t$$
$$y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + x_{4t} - \beta_3 x_{4t} + u_t$$

- Gather terms in β's together and rearrange

$$(y_t - x_{4t}) = \beta_1 + \beta_2 x_{2t} + \beta_3 (x_{3t} - x_{4t}) + u_t$$

- This is the restricted regression. We actually estimate it by creating two new variables, call them, say, P_t and Q_t.

$$P_t = y_t - x_{4t}$$
$$Q_t = x_{3t} - x_{4t}$$

so

$P_t = \beta_1 + \beta_2 x_{2t} + \beta_3 Q_t + u_t$ is the restricted regression we actually estimate.

Calculating the F-Test Statistic

- The test statistic is given by

$$test\ statistic = \frac{RRSS - URSS}{URSS} \times \frac{T - k}{m}$$

where $URSS$ = RSS from unrestricted regression

$RRSS$ = RSS from restricted regression

m = number of restrictions

T = number of observations

k = number of regressors in unrestricted regression

including a constant in the unrestricted regression (or the total number of parameters to be estimated).

The *F*-Distribution

- The test statistic follows the *F*-distribution, which has 2 d.f. parameters.
- The value of the degrees of freedom parameters are m and $(T-k)$ respectively (the order of the d.f. parameters is important).
- The appropriate critical value will be in column m, row $(T-k)$.
- The *F*-distribution has only positive values and is not symmetrical. We therefore only reject the null if the test statistic > critical *F*-value.

Determining the Number of Restrictions in an *F*-test

- Examples :

H_0: hypothesis	No. of restrictions, m
$\beta_1 + \beta_2 = 2$	1
$\beta_2 = 1$ and $\beta_3 = -1$	2
$\beta_2 = 0$, $\beta_3 = 0$ and $\beta_4 = 0$	3

- If the model is $y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 x_{4t} + u_t$, then the null hypothesis

  H_0: $\beta_2 = 0$, and $\beta_3 = 0$ and $\beta_4 = 0$ is tested by the regression *F*-statistic. It tests the null hypothesis that all of the coefficients except the intercept coefficient are zero.

- Note the form of the alternative hypothesis for all tests when more than one restriction is involved: H_1: $\beta_2 \neq 0$, or $\beta_3 \neq 0$ or $\beta_4 \neq 0$

What we Cannot Test with Either an *F* or a *t*-test

- We cannot test using this framework hypotheses which are not linear or which are multiplicative, e.g.

  $H_0: \beta_2 \beta_3 = 2$ or $H_0: \beta_2^{\,2} = 1$

  cannot be tested.

The Relationship between the *t* and the *F*-Distributions

- Any hypothesis which could be tested with a *t*-test could have been tested using an *F*-test, but not the other way around.

  For example, consider the hypothesis

  $H_0: \beta_2 = 0.5$

  $H_1: \beta_2 \neq 0.5$

  We could have tested this using the usual *t*-test: $test\ stat = \dfrac{\hat{\beta}_2 - 0.5}{SE(\hat{\beta}_2)}$

  or it could be tested in the framework above for the *F*-test.
- Note that the two tests always give the same result since the *t*-distribution is just a special case of the *F*-distribution.
- For example, if we have some random variable Z, and $Z \sim t(T\text{-}k)$ then also $Z^2 \sim F(1, T\text{-}k)$

F-test Example

- Question: Suppose a researcher wants to test whether the returns on a company stock (y) show unit sensitivity to two factors (factor x_2 and factor x_3) among three considered. The regression is carried out on 144 monthly observations. The regression is $y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 x_{4t} + u_t$
  - What are the restricted and unrestricted regressions?
  - If the two RSS are 436.1 and 397.2 respectively, perform the test.

- Solution:
  Unit sensitivity implies H_0: $\beta_2=1$ and $\beta_3=1$. The unrestricted regression is the one in the question. The restricted regression is $(y_t - x_{2t} - x_{3t}) = \beta_1 + \beta_4 x_{4t} + u_t$ or letting $z_t = y_t - x_{2t} - x_{3t}$, the restricted regression is $z_t = \beta_1 + \beta_4 x_{4t} + u_t$
  In the *F*-test formula, $T=144$, $k=4$, $m=2$, $RRSS=436.1$, $URSS=397.2$
  F-test statistic = 6.68. Critical value is an $F(2,140) = 3.07$ (5%) and 4.79 (1%).
  Conclusion: Reject H_0.

Data Mining

- Data mining is searching many series for statistical relationships without theoretical justification.
- For example, suppose we generate one dependent variable and twenty explanatory variables completely randomly and independently of each other.
- If we regress the dependent variable separately on each independent variable, on average one slope coefficient will be significant at 5%.
- If data mining occurs, the true significance level will be greater than the nominal significance level.

Goodness of Fit Statistics

- We would like some measure of how well our regression model actually fits the data.
- We have goodness of fit statistics to test this: i.e. how well the sample regression function (srf) fits the data.
- The most common goodness of fit statistic is known as R^2. One way to define R^2 is to say that it is the square of the correlation coefficient between y and $\hat{y}$.
- For another explanation, recall that what we are interested in doing is explaining the variability of y about its mean value, $\bar{y}$, i.e. the total sum of squares, *TSS*:

$$TSS = \sum_t (y_t - \bar{y})^2$$

- We can split the *TSS* into two parts, the part which we have explained (known as the explained sum of squares, *ESS*) and the part which we did not explain using the model (the *RSS*).

Defining R^2

- That is, $TSS = ESS + RSS$

$$\sum_t (y_t - \bar{y})^2 = \sum_t (\hat{y}_t - \bar{y})^2 + \sum_t \hat{u}_t^2$$

- Our goodness of fit statistic is

$$R^2 = \frac{ESS}{TSS}$$

- But since $TSS = ESS + RSS$, we can also write

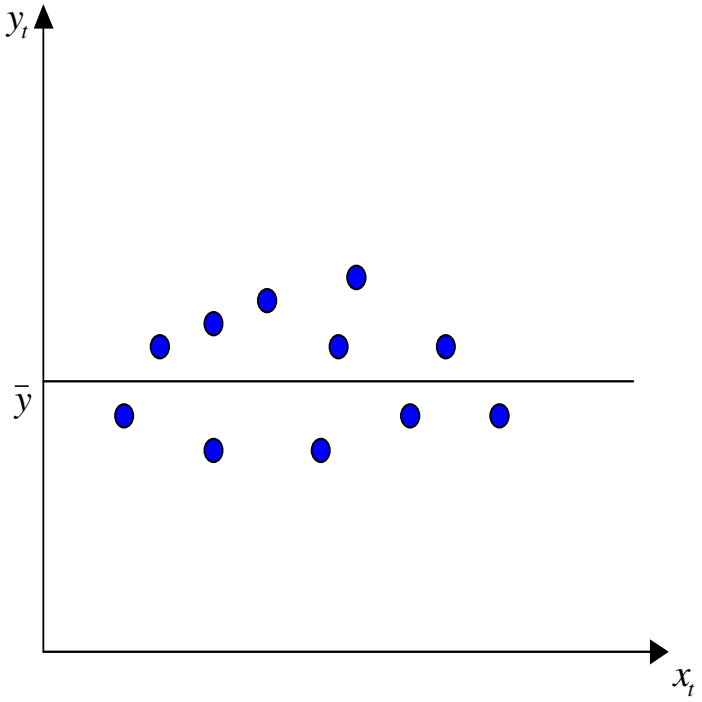
$$R^2 = \frac{ESS}{TSS} = \frac{TSS - RSS}{TSS} = 1 - \frac{RSS}{TSS}$$

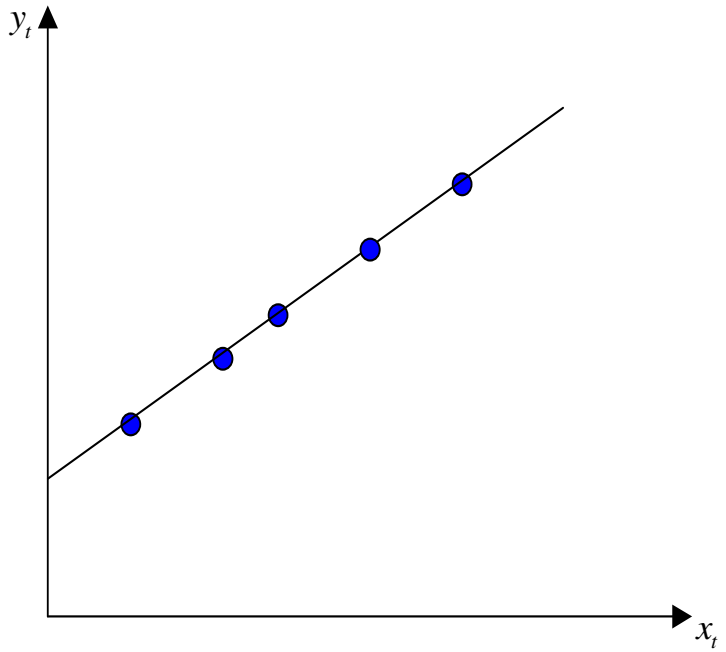
- R^2 must always lie between zero and one. To understand this, consider two extremes

$RSS = TSS$ i.e. $ESS = 0$ so $R^2 = ESS/TSS = 0$

$ESS = TSS$ i.e. $RSS = 0$ so $R^2 = ESS/TSS = 1$

The Limit Cases: $R^2 = 0$ and $R^2 = 1$





Problems with R^2 as a Goodness of Fit Measure

- There are a number of them:

1. R^2 is defined in terms of variation about the mean of y so that if a model is reparameterised (rearranged) and the dependent variable changes, R^2 will change.

2. R^2 never falls if more regressors are added. to the regression, e.g. consider:

Regression 1: $y_t = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + u_t$

Regression 2: $y = \beta_1 + \beta_2 x_{2t} + \beta_3 x_{3t} + \beta_4 x_{4t} + u_t$

R^2 will always be at least as high for regression 2 relative to regression 1.

3. R^2 quite often takes on values of 0.9 or higher for time series regressions.

Adjusted R^2

- In order to get around these problems, a modification is often made which takes into account the loss of degrees of freedom associated with adding extra variables. This is known as $\overline{R}^2$, or adjusted R^2:

$$\overline{R}^2 = 1 - \left[\frac{T-1}{T-k} (1 - R^2) \right]$$

- So if we add an extra regressor, k increases and unless R^2 increases by a more than offsetting amount, $\overline{R}^2$ will actually fall.

- There are still problems with the criterion:
  1. A "soft" rule
  2. No distribution for $\overline{R}^2$ or R^2

A Regression Example:
Hedonic House Pricing Models

- Hedonic models are used to value real assets, especially housing, and view the asset as representing a bundle of characteristics.
- Des Rosiers and Thérialt (1996) consider the effect of various amenities on rental values for buildings and apartments 5 sub-markets in the Quebec area of Canada.
- The rental value in Canadian Dollars per month (the dependent variable) is a function of 9 to 14 variables (depending on the area under consideration). The paper employs 1990 data, and for the Quebec City region, there are 13,378 observations, and the 12 explanatory variables are:

  LnAGE - log of the apparent age of the property

  NBROOMS - number of bedrooms

  AREABYRM - area per room (in square metres)

  ELEVATOR - a dummy variable = 1 if the building has an elevator; 0 otherwise

  BASEMENT - a dummy variable = 1 if the unit is located in a basement; 0 otherwise

Hedonic House Pricing Models: Variable Definitions

OUTPARK - number of outdoor parking spaces

INDPARK - number of indoor parking spaces

NOLEASE - a dummy variable = 1 if the unit has no lease attached to it; 0 otherwise

LnDISTCBD - log of the distance in kilometres to the central business district

SINGLPAR - percentage of single parent families in the area where the building stands

DSHOPCNTR- distance in kilometres to the nearest shopping centre

VACDIFF1 - vacancy difference between the building and the census figure

- Examine the signs and sizes of the coefficients.
  - The coefficient estimates themselves show the Canadian dollar rental price per month of each feature of the dwelling.

Hedonic House Price Results
Dependent Variable: Canadian Dollars per Month

Variable	Coefficient	*t*-ratio	*A priori* sign expected
Intercept	282.21	56.09	+
LnAGE	-53.10	-59.71	-
NBROOMS	48.47	104.81	+
AREABYRM	3.97	29.99	+
ELEVATOR	88.51	45.04	+
BASEMENT	-15.90	-11.32	-
OUTPARK	7.17	7.07	+
INDPARK	73.76	31.25	+
NOLEASE	-16.99	-7.62	-
LnDISTCBD	5.84	4.60	-
SINGLPAR	-4.27	-38.88	-
DSHOPCNTR	-10.04	-5.97	-
VACDIFF1	0.29	5.98	-

Notes: Adjusted R^2 = 0.65l; regression *F*-statistic = 2082.27. Source: Des Rosiers and Thérialt

Tests of Non-nested Hypotheses

- All of the hypothesis tests concluded thus far have been in the context of "nested" models.

- But what if we wanted to compare between the following models?

$$\text{Model 1}: y_t = \alpha_1 + \alpha_2 x_{2t} + u_t$$

$$\text{Model 2}: y_t = \beta_1 + \beta_2 x_{3t} + v_t$$

- We could use R^2 or adjusted R^2, but what if the number of explanatory variables were different across the 2 models?

- An alternative approach is an encompassing test, based on examination of the hybrid model: $\text{Model 3}: y_t = \gamma_1 + \gamma_2 x_{2t} + \gamma_3 x_{3t} + w_t$

Tests of Non-nested Hypotheses (cont'd)

- There are 4 possible outcomes when Model 3 is estimated:
  - γ_2 is significant but γ_3 is not
  - γ_3 is significant but γ_2 is not
  - γ_2 and γ_3 are both statistically significant
  - Neither γ_2 nor γ_3 are significant

- Problems with encompassing approach
  - Hybrid model may be meaningless
  - Possible high correlation between x_2 and x_3.

Mutliple Regressions: APT

1. Creating a workfile and importing data

   cd C:\data1

   workfile APT m 1986:3 2007:4

   read(B2,s=cpiai) macro.xls 13

2. Construct data

   *Genr rsandp=100*dlog(sandp)*

   *Genr rmicrosoft=100*dlog(microsoft)*

   Genr MUSTB3M=USTB3M/12

   Genr ersandp=rsandp - MUSTB3M

   Genr ermicrosoft=rmicrosoft - MUSTB3M

   Genr dspread = baa_aaa_spread01-baa_aaa_spread01(-1)

   Genr dcredit = consumer_credit0-consumer_credit0(-1)

   Genr dprod = industrial_produ-industrial_produ (-1)

   Genr dmoney = m1money_supply01 - m1money_supply01(-1)

Mutliple Regressions: APT (2)

*Genr inflation = 100*dlog(cpi)*

Genr term = ustb10y - ustb3m

Genr dinflation = inflation - inflation(-1)

Genr rterm = term- term(-1)

3. Estimate the model (consider F stat, t-stats and wald tests)

equation microsoft_APT1.ls ermicrosoft c ersandp dprod dcredit dinflation dmoney dspread rterm

microsoft_APT1.wald c(3)=0, c(4)=0, c(7)=0

equation microsoft_APT2.ls ermicrosoft c ersandp dinflation dmoney rterm

microsoft_APT2.wald c(3)=0

equation microsoft_APT3.ls ermicrosoft c ersandp dmoney rterm